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AUTOCORRELATION, INVESTMENT HORIZON AND EFFICIENT FRONTIER COMPOSITION

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Summary:

This paper demonstrates that modest amounts of security autocorrelation can cause the composition of an efficient frontier constructed on the basis of one holding period length assumption to differ substantially from the composition of an efficient frontier constructed on the basis of another holding period length assumption. This will be true even if the efficient frontiers are constructed using the same data base over the same total time span. This finding is derived mathematically and demonstrated empirically. The paper discusses the serious consequences of this finding.

AUTOCORRELATION, INVESTMENT HORIZON AND EFFICIENT FRONTIER COMPOSITION

I. Introduction

In 1949 Working developed the time/variance test of the random walk model. He pointed out that if security returns are assumed to be stationary and independent, the variance of returns will increase in (approximately) direct proportion to the length of the differencing interval over which returns are measured.

In 1977 Schwartz and Whitcomb derived equations describing the magnitude of the effect that autocorrelation has on the time/variance relationship.[1] They demonstrated that the variances of securities with negative autocorrelations, increase less than proportionally with increasing measurement interval, while variances of positively autocorrelated securities increase more than proportionally.

This paper will demonstrate that this phenomenon has implications far beyond its role as a test of the random walk model.

We will show that changes in relative variance due to changes in the length of the measurement interval can cause a portfolio which is low return and low risk (and efficient) for one holding period length assumption to be low return and high risk (and inefficient) for a different holding period length.[2]

We will show that these intervaling effects can cause an efficient frontier appropriate to one holding period length assumption to be substantially and systematically different from an efficient frontier appropriate to another holding period length assumption (see Appendix II for a brief description of the holding period problem).[3]

Holding period length dependent changes are particularly important when we consider that (in the author's opinion) the holding period assumption implicit in most empirical research is nothing more than the intervaling length of a conveniently available data base. Therefore, the systematic and substantial changes in efficient frontier composition demonstrated in this paper suggest:

(1) Practitioners may make inappropriate investment decisions when the interval of a conveniently available data base (e.g., the daily CRSP tape) does not correspond to the length of time for which the practitioner actually wishes to hold his investment.

(2) Empirical tests of the efficient frontier may reveal characteristics which only relate to holding period lengths similar to the length implicit in the test. A different holding period length assumption might produce different characteristics. For example, in the Implications Section of this paper this property will be used in an attempt to reconcile the conflicting empirical evidence on the value of fixed income securities in efficient portfolios (e.g., see Norgard (1974) or Alexander (1977) or Sarnat (1974)).

The magnitude of the effect of the holding period length assumption on the efficient frontier may surprise some readers. The fact that there is some effect should be less of a surprise. In the last decade and a half, a number of authors have documented a number of financial concepts whose characteristics change depending on the length of the intervaling assumption used in their observation.

In 1965 Tobin demonstrated that under the assumptions of stationarity and independence the efficient frontier changes shape from one holding

period length assumption to another. Gressis, Philippatos and Hayya (1976) have suggested that this change in shape causes the capital market line to intersect the efficient frontier at different portfolios for different holding period length assumptions.

In 1969 Jensen pointed out that the capital market relationship between beta and expected return can only be linear for one specific holding period length assumption.

Levy (1972) has demonstrated that Sharpe's (1966) return to variability performance measure changes from one holding period length assumption to another. Levhari and Levy (1977) demonstrated a similar bias in Treynor's (1965) reward to volatility measure.

Levhari and Levy (1977) have also demonstrated that the magnitude of beta depends on the length of the intervaling assumption used in its measurement. This finding has been empirically confirmed by Smith (1978). Hasty and Fielitz (1975) have demonstrated that even the ranking of systematic risk measures is sensitive to the holding period length assumption.

The concept that the length of the holding period assumption can effect the measurement of security characteristics is not new. However, our use of autocorrelation in analysing these changes is new, as is our derivation of the effects of autocorrelation and holding period length on the efficient frontier.

This paper will be organized as follows: Section I introduces the basic problem. Section II analyses the theoretical effects of autocorrelation and holding period length on security variances and covariances. Section III describes the data used in our empirical tests. Section IV presents empirical tests of the effects of autocorrelation and holding

period on security variance. The effects are seen to be substantial and well explained by our theories. Section V empirically tests the effect of autocorrelation and holding period length on efficient frontier composition and performance. Again, the results are substantial and well explained by our theories. Section VI describes some of the implications of our results. Two particularly important implications are: First, many practitioners are probably using efficient frontiers unsuited to their actual holding period length requirements. Second, empirical tests of the efficient frontier may discover properties which only exist for the specific holding period length implicit in the tests. Section VII summarizes our work.

II. Holding Period and Autocorrelation (Theoretical Framework)

Tobin (1965) seems to have been the first to analyze the effect that the holding period length assumption has on the efficient frontier concept. Tobin used the assumptions of stationarity and independence to derive equations for N period expected return, ER_N, and Variance, σ_N^2; given single period expected return, Er_1, and variance, σ_1^2.

$$ER_N = (1 + Er_1)^N - 1 \tag{1}$$

$$\sigma_N^2 = [\sigma_1^2 + (1 + Er_1)^2]^N - (1 + Er_1)^{2N} \tag{2}$$

These equations have an important implication. Equation (2) makes it clear that the minimum variance portfolio at any level of single period expected return will also be the minimum variance portfolio at the corresponding level of N period expected return. Therefore a portfolio which

is on the efficient frontier for one holding period length assumption will also be on the efficient frontier for any longer holding period.[4]

We will now relax Tobin's assumptions of stationarity and independence and derive equations analogous to equations (1) and (2) under the assumptions of stationarity and autocorrelation.

Schwartz and Whitcomb (1977) have already provided us with a description of the effect of the length of the holding period assumption on security variance under the assumptions of stationarity and autocorrelation.

Following Schwartz and Whitcomb we express N period (log) returns, R_N, as a sum of single period (log) returns, R_i.[5]

$$R_N = \sum_{i=1}^{N} R_i \qquad (3)$$

We can use the formula for the variance of a sum to express the variance, σ_N^2, of N period returns as:

$$\sigma_N^2 = \sum_{u=1}^{N} \sum_{v=1}^{N} \mathrm{Cov}(R_u, R_v). \qquad (4)$$

Stationarity implies that:

$$\mathrm{Cov}_{j,k} = \mathrm{Cov}_{m,n}$$

for: $k = j + i$ and $n = m + i$

for all integer values of i and all positive integer values of j and m for which k and n are also positive integers. We can therefore express equation (4) as:

$$\sigma_N^2 = N\sigma_{j,j}^2 + \sum_{i=1}^{N-1} (N-i)(2)(\mathrm{Cov}_{j,j+i}) \qquad (5)$$

where j is the positive integer arbitrarily assigned to the first of our N single period time intervals.

We can isolate the specific effect autocorrelation has on security variance by defining a new variable: "variance distortion," VD_N. We will define variance distortion as: The difference (expressed as a percentage) between the N period variance predicted from single period variance under the assumptions of stationarity and <u>autocorrelation</u>; and the N period variance predicted from single period variance under the assumptions of stationarity and <u>independence</u>. Variance distortion can be expressed as follows:

$$VD_N = 100(N\sigma^2_{j,j} + \sum_{i=1}^{N-1} [(N-i)(2)(Cov_{j,j+i})] - N\sigma^2_{j,j})/N\sigma^2_{j,j} \quad (6)$$

$$= 100(\sum_{i=1}^{N-1} [(N-i)(2)(Cov_{j,j+i})]/N\sigma^2_{j,j}) \quad (7)$$

where the factor of 100 has been used to convert the fraction into a percentage.

If we substitute

$$Cov_{j,j+i} = \rho_{j,j+i}\, \sigma_{j,j}\, \sigma_{j+i,j+i}$$

into equation (7), we have:

$$VD_N = 200 [\sum_{i=1}^{N-1} (N-i)\, \rho_{j,j+i}\, \sigma_{j,j}\, \sigma_{j+i,j+i}]/N\sigma^2_{j,j}$$

The assumption of stationarity gives us:

$$VD_N = 200 [\sum_{i=1}^{N-1} (N-i)\, \rho_{j,j+i}]/N \quad (8)$$

where the $\rho_{j,j+i}$ term is the autocorrelation coefficient of our single period (log) returns.

If all autocorrelation effects are of a low order (i.e., $\rho_{j,j+i} = 0$ for all $i>m$ where m is a small positive integer), we can then conclude that as the holding period becomes very large relative to m (i.e., as $N\to\infty$), we can rewrite equation (8) as follows:

$$\lim_{N\to\infty}(VD_N) = 200 \sum_{i=1}^{m} (\rho_{j,j+i}) \tag{9}$$

Equations (8) and (9) have some interesting implications. From equation (8) we can see that, in general, as N increases, variance distortion also increases. Equation (8) also shows that if security autocorrelation is limited to the first few lag intervals, as N increases, VD_N increases and asymptotically approaches the fixed value indicated by equation (9). Equations (8) and (9) also suggest that relatively small amounts of autocorrelation can produce substantial levels of variance distortion.

The expected value of an N period (log) return under the assumptions of stationarity and autocorrelation is relatively easy to derive. We need only apply an expectation operation, E, to both sides of equation (3)

$$ER_N = E \sum_{i=1}^{N} R_i$$

The expectation of a sum is equal to the sum of the expectations:

$$ER_N = \sum_{i=1}^{N} ER_i \tag{10}$$

Equation (10) shows that autocorrelation has no affect on N period expected (log) return.

Equation (8) indicates that negatively (positively) autocorrelated securities will have lower and lower (higher and higher) variances relative to serially uncorrelated securities as the holding period assumption is lengthened.

Equation (10) indicates that autocorrelation has no effect on N period expected return. We would therefore expect negatively (positively) autocorrelated securities to become increasingly (decreasingly) attractive candidates for inclusion in the efficient frontier as the holding period is lengthened.

An efficient frontier portfolio rarely consists of a single security. It is therefore necessary to analyse what kinds of securities (in terms of autocorrelation) will combine to produce a portfolio with increasing or decreasing relative variance as the holding period is lengthened.

In equation (4) we described the N period variance of a single security as the variance of the sum of N single period returns. We can therefore describe the N period variance of a sum of P different securities as:

$$\sigma^2_{P,N} = \sum_{h=1}^{P} \sum_{k=1}^{P} \sum_{u=1}^{N} \sum_{v=1}^{N} \mathrm{Cov}(R_{k,u}, R_{h,v}) \tag{11}$$

where the first index of R (i.e., k or h) identifies individual securities and the second index of R (i.e., u or v) identifies individual, single period time intervals.

Equation (11) can be broken down into two components:

$$\sigma^2_{P,N} = \sum_{k=1}^{P} \sum_{u=1}^{N} \sum_{v=1}^{N} \mathrm{Cov}(R_{k,u}, R_{k,v})$$

$$+ \sum_{h=1}^{P} \sum_{k=1}^{P} \sum_{u=1}^{N} \sum_{v=1}^{N} \mathrm{Cov}(R_{k,u}, R_{h,v}) \quad \text{where } k \neq h \qquad (12)$$

The first component of our N period portfolio variance is nothing more than the sum of the N period variances of our P securities. Equation (8) suggests that this term will become relatively smaller (larger) with increased N if the portfolio largely consists of negatively (positively) autocorrelated securities.

The second element of $\sigma^2_{P,N}$ consists of a variety of intersecurity covariances most of which are both intersecurity (i.e., $k \neq h$) and intertemporal (i.e., $u \neq v$). If we can assume these terms are not systematically affected by autocorrelation we are left with the intuitively satisfying conclusion that portfolios with relatively declining (increasing) relative variances consist primarily of securities with relatively declining (increasing) variances.[6]

These relatively declining (increasing) variance securities are identified in equation (8) as negatively (positively) autocorrelated securities. Therefore, as the holding period assumption is lengthened efficient frontier portfolios will contain more and more negatively autocorrelated securities and less and less positively autocorrelated securities.

III. The Data

The data we used in this research consisted of daily return data from January 31, 1975, to February 19, 1976, for 54 common stocks, 15

convertible preferred stocks, 7 convertible bonds and 2 warrants. The actual securities used in this study are listed in Appendix 1.[7]

In our study we varied the length of the holding period assumption while keeping the span of the data base constant. We used our daily return data base to calculate returns of 1, 4, 8, 12, and 24 day holding periods using, in each case, data spanning the 264 day period, January 31, 1975, to February 19, 1976.

All of our computations for all of our holding period length assumptions will be based upon the exact same 265 days of data. We will change the holding period assumption by changing the way in which we group the data. For example, our one day holding period computations will be based upon 264 one day interval returns; our 24 day holding period computations will be based upon eleven, 24 day interval returns.

Equation (8) suggests that N period security variance is a positive function of single period autocorrelation calculated over a variety of lag intervals. We therefore calculated one day autocorrelation coefficients for lag intervals of one through sixteen days.

Our data base contains 78 securities. If all of these securities are serially independent, the null hypothesis of serial independence will be rejected at the 5 percent level for about 4 securities. We therefore judged security autocorrelation to be significant <u>only</u> if it resulted in substantially more than four securities showing significant autocorrelation.

By this criteria we discovered three types of significant autocorrelation. Thirty of our first order, thirteen of our second order,

and seven of our third order serial correlation coefficients between daily returns were significant at the 5 percent level.[8]

We will need to identify and single out for further study those securities exhibiting an unambiguously large amount of autocorrelation.

Our criteria is a necessarily arbitrary formulation involving all three types of autocorrelation which we found to be significant. Specifically, we will consider a security to exhibit substantial autocorrelation if the sum of the one day return autocorrelations for lags one, two, and three days has an absolute value greater than or equal to three times the standard error for one day holding period, one day lag, autocorrelation (i.e., the absolute value of the sum must be greater than .18).

This criteria selected 8 securities as having substantial negative autocorrelation and 10 securities as having substantial positive autocorrelation. These 18 securities (23 percent of our original data base) are indicated in Appendix 1.[9]

These securities are not intended to represent a unique or exhaustive list of autocorrelated securities. A different criteria might have selected a different group of securities. These securities seem to exhibit an unambiguously high level of autocorrelation and they will prove adequate for our purposes.

IV. Variance as a Function of Holding Period Length (Empirical Tests)

Equations (8) and (9) suggest that as we lengthen the holding period assumption, security variance becomes a positive function of security autocorrelation. In order to test this theory, we will first attempt to measure

the extent to which the Tobin (non-autocorrelation) model of security variance (equation (2)) is inaccurate.

We used the Tobin model to predict, on the basis of measured one day expected returns and variances, the variance which should exist in the 4, 8, 12, and 24 day returns for each of our 18 autocorrelated securities. We then calculated the actual variance of each security for these same holding periods.

We then computed the percent by which the Tobin model deviates from actual measured variance. This percent difference will be referred to as actual variance distortion, $VD_{A,N}$ and will be calculated as:

$$VD_{A,N} = 100(V_{A,N} - V_{T,N})/V_{T,N}$$

where $V_{A,N}$ is the actual measured variance for an N day holding period and $V_{T,N}$ is the Tobin model prediction of N day variance.

The first and third rows of data in Table 1 contain the average actual variance distortion for our eight negatively autocorrelated securities and our ten positively autocorrelated securities respectively. As these rows indicate, the deviations of the actual variances from the Tobin model predictions are substantial.

The second and fourth rows of Table 1 contain our own predictions of variance distortion. These results were calculated from equation (8) using the serial correlation coefficient between one day returns for one and two day lag intervals.

As the reader can verify for himself, not only does actual security variance from one holding period to another differ drastically from the

TABLE 1

AVERAGE VARIANCE DISTORTION (Percent)

	Holding Period Length (Days)			
	4	8	12	24
Negatively Autocorrelated:				
Actual	-23.957	-25.020	-36.105	-41.294
Predicted	-19.717	-26.516	-28.783	-31.049
Positively Autocorrelated:				
Actual	38.072	50.414	31.573	30.796
Predicted	29.385	35.022	36.901	38.780

Tobin model, our own model predicts this deviation with remarkable accuracy.[10]

Place Table 1 about here

Given the magnitude and importance of our variance distortion findings, a statistical test seems to be in order. We therefore made use of a test of the time/variance relationship first presented by Young (1971).[11]

We first used the Young methodology to test the null hypothesis that the 24 day variances of each of our 18 securities can be predicted from one day means and variances using Tobin's time variance equation (2). This null hypothesis was rejected at the 5 percent significance level for 13 of our 18 securities.

We then used the Young methodology to test the null hypothesis that 24 day holding period security variance can be predicted using our own autocorrelation model as described in equation (8). We calculated equation (8) using daily return autocorrelation for lags of one and two days. Our model was rejected at the 5 percent level of significance for only 5 of our 18 securities. Clearly, more securities were rejected using our model than can be explained by random chance. Nevertheless, our model does seem to be a striking improvement over the Tobin model.

V. Efficient Frontier Composition as a Function of Holding Period Length (Empirical Tests)

We will now empirically test the relationship between efficient frontier composition and the holding period assumption. We are particularly interested in two aspects of this relationship.

TABLE 2 PART 1

EFFICIENT FRONTIER COMPOSITION
(A COMPARISON OF EXPECTED RETURNS, ACTUAL VARIANCES AND PROJECTED VARIANCES)

	HOLDING PERIOD LENGTH (DAYS)							
	1 DAY		4 DAY			8 DAY		
Portfolio Number	Return (x10)	Opt Var(x10)	Return (x10)	Bench Var(x10)	Opt Var(x10)	Return (x10)	Bench Var(x10)	Opt Var(x10)
1 (Maximum Return)	.037452	.019145	.151939	.081855	.081855	.281014	.112949	.112950
2	.033707	.003505	.137137	.018624	.018509	.271295	.033153	.031848
3	.029961	.001625	.121388	.010339	.009144	.244294	.020922	.018636
4	.026216	.001050	.106137	.006697	.005924	.213649	.014412	.012087
5	.022471	.000697	.090751	.004432	.003851	.182547	.009964	.008006
6	.018726	.000459	.075383	.002940	.002469	.151600	.007237	.005233
7	.014981	.000312	.060155	.002077	.001631	.120871	.005276	.003380
8	.011236	.000232	.044931	.001506	.001150	.090184	.003759	.002260
Minimum Variance	.009189	.000217	.037750		.001056	.076372		.002077

TABLE 2 PART 2

EFFICIENT FRONTIER COMPOSITION
(A COMPARISON OF EXPECTED RETURNS, ACTUAL VARIANCES
AND PROJECTED VARIANCES)

	HOLDING PERIOD LENGTH (DAYS)					
	12 DAY			24 DAY		
Portfolio Number	Return (x10)	Bench Var(x10)	Opt Var(x10)	Return (x10)	Bench Var(x10)	Opt Var(x10)
1 (Maximum) Return)	.475123	.292269	.292270	.827900	.310216	.310220
2	.413254	.046483	.045904	.810682	.080265	.075495
3	.363272	.026606	.022379	.734273	.061982	.045454
4	.317912	.018076	.013365	.643477	.045050	.026149
5	.271915	.012607	.007991	.550527	.033486	.015140
6	.226243	.009350	.004881	.457754	.025850	.008338
7	.180664	.006911	.003109	.365289	.019986	.004820
8	.135079	.005097	.002310	.272812	.015411	*
Minimum Variance	.115142		.002210	.289787		.003998

*STATED LEVEL OF RETURN IS NO LONGER ON THE EFFICIENT FRONTIER

TABLE 3

EFFICIENT FRONTIER COMPOSITION

Positively (Negatively) Autocorrelated
Securities as a Percent of the Total

Portfolio Number	HOLDING PERIOD LENGTH (DAYS) 1	4	8	12	24
1 (Maximum)	.0000 (.0000)	.0000 (.0000)	.0000 (.0000)	.0000 (.0000)	.0000 (.0000)
2	.0000 (.0000)	.2267 (.0003)	.2444 (.0000)	.2661 (.0000)	.0000 (.0000)
3	.2326 (.0893)	.0000 (.1264)	.2923 (.0631)	.0000 (.0558)	.0000 (.1126)
4	.1682 (.0604)	.0000 (.1304)	.2332 (.1862)	.0000 (.2975)	.0000 (.5074)
5	.1186 (.1413)	.0000 (.1741)	.1413 (.2038)	.0378 (.3764)	.0000 (.5488)
6	.1711 (.2158)	.0000 (.4126)	.0834 (.3577)	.0304 (.5379)	.0000 (.5139)
7	.2331 (.2197)	.0753 (.5166)	.0129 (.3697)	.0046 (.6210)	.0000 (.6006)
8	.2647 (.2142)	.1137 (.4586)	.0713 (.2859)	.0000 (.6105)	*
Minimum Variance	.2984 (.1906)	.1373 (.4025)	.0966 (.2417)	.0000 (.6013)	.0000 (.4650)

*EFFICIENT PORTFOLIO DOES NOT EXIST AT THIS LEVEL OF RETURN

First. Are changes in efficient frontier composition consistent with the autocorrelation model we derived in equations (3) through (12)?

Second. What is the loss in efficiency (i.e., the difference between realized and optimum variance at the same level of expected return) when the performance of an efficient frontier constructed on the basis of one holding period length assumption is measured over a different holding period length?

We began our calculations by computing an efficient frontier based on the one day holding period expected returns, variances and covariances of our 78 security data base.[12] This procedure produced the eight efficient frontier portfolios whose expected returns and variances are described in the first eight rows of the first two columns of Table 2 and whose compositions are described in the first eight rows of the first column of Table 3.

Place Tables 2 and 3 about here

If Tobin's (1965) assumptions of stationarity and independence are correct, efficient frontier composition does not change from one holding period assumption to another. Therefore, under Tobin's assumptions, these eight one day holding period efficient frontier portfolios should be part of any efficient frontier based upon any holding period assumption.

These eight one day holding period portfolios will therefore serve as "benchmark" portfolios against which holding period induced changes in efficient frontier composition will be measured. For each of these eight benchmark portfolios we computed the returns and variances which would accrue to these portfolios if they were held for 4, 8, 12, and 24 day

periods. These expected returns and variances are described in the first 8 rows under the "RETURN" and "BENCH VAR" columns for each of the four long holding period assumptions listed in Table 2.[13]

The reader should keep in mind that, of these five sets of benchmark variances (one set for each of five holding period assumptions) only the one day holding period variances are the direct result of an optimizing algorithm. The variances computed for the four longer holding period assumptions will only (normally) be optimum if the efficient frontier does not change when the holding period length assumption is altered.

We will now compute actual long holding period efficient frontiers against which to compare our benchmark portfolios. In order to make our long holding period efficient portfolios correspond to our benchmark portfolios, we will, for each holding period assumption, calculate efficient frontier portfolios whose levels of expected return are exactly those of our benchmark portfolios. In this manner we will have derived, for each of our long holding period assumptions, two sets of variances at each of our eight levels of benchmark expected return. The variances of these optimum portfolios are listed under the heading "OPT VAR" for each of the longer holding periods described in Table 2. The composition of these optimum portfolios is described in columns 2 through 5 of Table 3.

The results of our computations are very interesting. If Tobin's assumptions of stationarity and independence are correct then efficient frontier composition is independent of holding period and all five columns of Table 3 should be identical. The reader can easily observe that this is not the case. Let us attempt to understand these changes.

Our theoretical results of equation (8) and our empirical results of Table 1 both suggest that as the holding period assumption is lengthened, security variance is a large magnitude positive function of security autocorrelation. This functional relationship suggests that as the holding period assumption is lengthened, the variances of negatively (positively) autocorrelated securities increase less (more) rapidly than the variances of uncorrelated securities. This relative decrease (increase) in risk should cause negatively (positively) autocorrelated securities to become a larger and larger (smaller and smaller) proportion of the efficient frontier as the holding period assumption is lengthened.

This theoretical relationship is demonstrated empirically in Table 3. Each element of the Table gives us the actual composition (in terms of autocorrelation) of the portfolios which produced the "RETURN" and "OPT VAR" columns for the corresponding portfolio number and holding period listed in Table 2.

The reader will note that in general the percentage of negatively (positively) autocorrelated securities in these efficient portfolios increases (decreases) steadily as the holding period increases. For example, our one day holding period efficient frontier portfolios contain an average of 16.52% positively autocorrelated securities while our 24 day holding period efficient frontier portfolios contain none. Conversely, our one day holding period efficient portfolios contain an average of 12.57% negatively autocorrelated securities while our 24 day efficient frontier portfolios contain an average of 34.35%.

Clearly, efficient frontier composition does change from one holding period length assumption to another.[14] Table 3 illustrates the practical

importance of these changes. If there were no change in efficient frontier composition from one holding period assumption to another, benchmark ("BENCH VAR") and optimum ("OPT VAR") columns for any one holding period assumption would be identical (the actual portfolios themselves would be identical). The difference between these columns describes the loss in efficiency which occurs when an efficient frontier constructed on the basis of a one day holding period assumption is actually measured over a longer time interval.

For example, our 24 day holding period computations show that several of the 24 day variances projected from our one day holding period efficient portfolios are more than twice the actual minimum variances attainable at the stated levels of expected return.[15] Furthermore, for portfolio number eight there is no efficient portfolio corresponding to the 24 day benchmark expected return levels projected from our one day holding period efficient portfolios. This return level is entirely off of the efficient frontier. For example, for portfolio number eight the optimum 24 day efficient frontier not only offers a substantially lower variance than the variance projected from our one day holding period efficient portfolio, the optimum efficient frontier offers this lower variance at a higher level of expected return (i.e., the minimum variance portfolio offers a 24 day variance of .0003998 at a 24 day expected return of .0289787, while the one day holding period optimum portfolio offers a 24 day variance of .0015411 at a .0272812 level of 24 day expected return).

VI. Implications

Tables 2 and 3 describe our most important empirical findings. These tables suggest that as the holding period assumption is changed, the

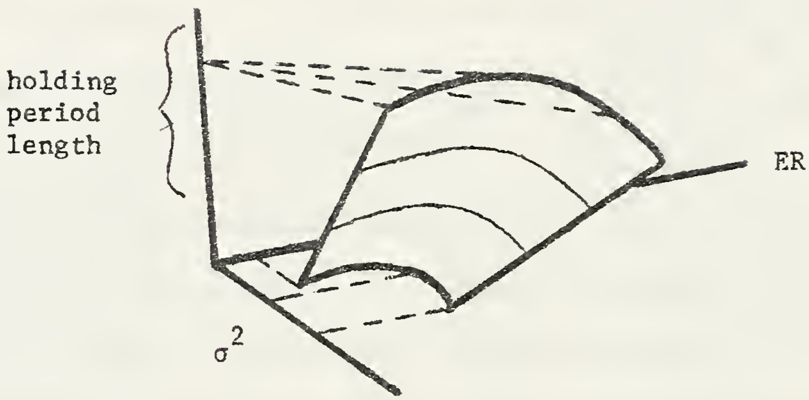

FIGURE 1

Three Dimensional Efficient Frontier

efficient frontier may change its composition, efficiency, and shape. These findings add a new dimension (literally) to the efficient frontier concept.

If the characteristics of the efficient frontier change from one holding period assumption to another, a general model of the efficient frontier concept must be defined in three dimensions (expected return, variance, and holding period length). This concept is illustrated in Figure 1.

Place Figure 1 about here

The three dimensional curved surface illustrated in Figure 1 represents the locus of all efficient frontiers for all possible holding period length assumptions. We will call this surface the efficient frontier surface. Clearly, any efficient frontier research which is conducted on the basis of one single holding period assumption cannot describe the entire efficient frontier surface. Research conducted on the basis of a single holding period assumption only succeeds in describing the efficient frontier <u>line segment</u> which is the intersection of the efficient frontier surface and a place perpendicular to the holding period length axis intersecting the axis at a point describing the holding period length specified in the study. This slice of the efficient frontier surface will not necessarily be representative of the surface as a whole. Therefore, if the efficient frontier changes characteristics from one holding period assumption to another, empirical research based upon a single holding period assumption <u>may</u> not apply to other possible holding period assumptions. This finding has two particularly interesting implications:

First, when historical data is used to construct an efficient frontier, the holding period assumption implicit in the frontier is usually

the intervaling assumption (daily data? monthly data?) of the most conveniently available data base. An efficient frontier whose holding period assumption is based upon data availability may bear little resemblance to the efficient frontier relevant to a specific investor's holding period requirement. The daily and monthly CRSP Tapes are two of the most widely used data bases in financial research. Table 3 clearly illustrates how little resemblance there is between an efficient frontier based on daily data and efficient frontiers based on monthly (actually 24 day) data.

Second, equations (3) through (12) suggest that a class of securities which is characterized by large levels of autocorrelation of a specific sign (e.g., warrants; see Leabo and Rogalski (1975)) may be overrepresented in an efficient frontier relevant to one holding period length assumption and underrepresented in a frontier appropriate to another holding period. This phenomenon may explain the contradictory results of various tests of the value of fixed income securities in portfolio diversification. A comparison of Alexander (1977) and Sarnat (1974) provides a good example of the controversy. Both authors use common stock and fixed income security data to construct efficient frontiers. Alexander concludes that fixed income securities provide a valuable addition to the efficient frontier, but Sarnat concludes they are of little value (for example, at a 12.4% annual expected return Alexander's efficient frontier is 37.55% government bonds while at a 12.3% expected return Sarnat's efficient portfolio has no fixed incomes). This paper provides a possible explanation for the conflict. Alexander uses quarterly data. Sarnat uses annual data. Given the positive autocorrelation often found in fixed income security interest

rates and returns (e.g., Fand (1966)) equations (3) through (12) indicate that the variances and covariances of fixed income securities probably increase faster as the holding period assumption is lengthened than securities with less autocorrelation. It is therefore possible that both Alexander and Sarnat are correct. For a quarterly holding period, fixed incomes may be quite attractive, but for an annual holding period assumption the variances and covariances may have increased so much faster than the risk measures of other securities that they are no longer attractive.[16,17]

VII. Summary

Our empirical results demonstrate that the composition of an efficient frontier constructed on the basis of one holding period length assumption can differ dramatically from the composition of an efficient frontier constructed on the basis of a different holding period length assumption. Our results also demonstrate that an efficient frontier constructed on the basis of one holding period length assumption can be extremely inefficient when evaluated over a different length holding period.

We have presented a theoretical model that suggests that these holding period induced changes in efficient frontier composition and efficiency are largely due to low order security autocorrelation. We have also presented several empirical tests which support this hypothesis. Section VI has analysed the important implications of these results.

In a sense, our autocorrelation model is a logical extension of existing portfolio theory. Portfolio theory has always emphasized the importance of the correlation between the returns of different securities. Our work merely suggests that the correlation between successive returns of the same security may also be important.

Appendix I (Part 1)

SERIAL CORRELATION COEFFICIENTS
FOR DAILY RETURNS FOR LAG
INTERVALS K=1-5

	LAG INTERVALS (DAYS)				
CONVERTIBLE PREFERRED:	1	2	3	4	5
AMER. HOME PROD. $2.00	0.018	0.034	-.041	-.052	0.008
A.T.&T. $4.00	0.021	0.042	0.053	-.134	-.058
ATLANTIC RICHFIELD $3.00	0.076	-.006	-.007	-.077	-.068
ATLANTIC RICHFIELD $2.80	0.252	-.078	-.070	-.111	-.110
GEN. TEL.& ELECT. $2.50 **	-.166	-.045	-.050	0.025	0.016
I.T.&T. SERIES E $4.00 **	-.146	-.060	-.095	0.090	-.001
I.T.&T. SERIES F $4.00 **	-.148	-.002	-.042	-.064	0.022
I.T.&T. SERIES H $4.00	0.078	-.034	-.028	0.035	0.062
I.T.&T. SERIES J $4.00	0.092	-.062	0.075	0.036	0.077
I.T.&T. SERIES K $4.00	-.007	0.055	-.061	0.094	-.028
I.T.&T. $5.00 *	0.069	0.046	0.118	0.110	0.084
I.T.&T. $2.25 *	0.136	-.011	0.079	0.035	-.023
I.T.&T. $4.50	-.079	0.101	0.075	0.067	0.012
MONSANTO $2.75	0.017	0.054	0.037	-.011	-.022
R.C.A. $4.00	-.029	-.057	0.032	0.031	-.098
WARRANTS:					
GULF & WESTERN	-.061	-.004	0.118	-.056	0.034
LOEW'S CORP.	0.057	-.063	0.003	-.010	-.083
CONVERTIBLE BONDS:					
CHASE MANHATTAN '96 6.5%	0.045	-.109	-.033	-.035	0.075
FED. NAT. MORT.'96 4.375%	0.041	-.027	0.143	-.089	-.008
GEN. TEL.& ELECT.'96 6.25% *	0.085	0.102	0.061	0.015	-.060
GULF & WESTERN '93 5.5%	0.002	-.131	0.127	0.025	-.049
KRESGE '99 6.0%	0.081	-.074	0.040	0.045	0.017
R.C.A.'92 4.5% **	-.295	-.050	0.051	0.028	-.045
XEROX '95 6.0%	-.004	-.089	0.096	0.011	-.016

* Positively autocorrelated
** Negatively autocorrelated

Appendix I (Part 2)

SERIAL CORRELATION COEFFICIENTS
FOR DAILY RETURNS FOR LAG
INTERVALS K=1-5

COMMON STOCK:	LAG INTERVALS (DAYS) 1	2	3	4	5
ALCOA *	0.312	0.070	-.047	-.026	0.014
AMERICAN CYANAMID	0.080	-.040	-.012	-.057	0.043
AMERICAN HOME PRODUCTS *	0.214	0.064	-.092	-.088	-.021
A.T.&T.	0.066	0.058	-.020	-.080	-.000
ATLANTIC RICHFIELD	0.187	-.021	-.009	-.117	-.099
AVON	0.116	-.086	0.007	-.008	0.004
BETHLEHEM STEEL	0.172	-.093	0.068	-.011	-.136
BRUNSWICK	-.012	-.070	0.039	0.060	0.129
BURROUGHS	0.141	-.126	0.011	-.010	0.011
CHASE MANHATTAN **	0.158	-.190	-.149	-.129	0.034
CITICORP	0.145	0.013	-.111	0.014	0.037
JOHN DEERE	0.104	-.039	0.001	0.011	-.064
DELTA AIRLINES	0.083	-.071	-.022	-.024	0.001
DIGITAL EQUIPMENT CORP.	0.094	-.050	0.040	-.058	-.052
DOW CHEMICAL	0.179	-.125	0.016	-.027	-.141
DU PONT *	0.241	-.000	0.045	0.026	-.000
EASTMAN KODAK	0.083	-.120	-.016	-.023	0.034
EXXON	0.156	-.075	-.042	-.025	-.076
FEDERAL NATIONAL MORTGAGE	0.075	-.075	0.091	-.068	-.010
FORD MOTOR	0.086	0.048	-.003	-.084	-.044
GENERAL ELECTRIC **	0.120	-.218	-.116	0.131	0.119
GENERAL MOTORS **	0.064	-.171	-.119	0.079	0.085
GEN. TEL. & ELECT.	0.171	-.061	0.059	0.111	-.102
GILLETTE	0.164	-.162	-.128	0.042	0.143
GOODYEAR	0.074	-.051	-.121	-.025	0.066
GULF OIL	0.044	-.010	0.011	-.023	-.001
I.N.A. CORP. *	0.184	0.027	0.027	0.028	0.094
I.B.M.	-.047	-.097	0.049	0.006	0.058
INTERNATIONAL HARVESTER	0.195	-.050	-.015	-.081	0.003
INTERNATIONAL PAPER	0.195	-.006	-.095	0.012	-.057
I.T.&T. *	0.134	-.022	0.084	0.090	-.047
KENNECOTT COPPER	0.101	-.032	-.018	-.046	-.071
KERR-MCGEE	0.101	0.021	-.012	0.032	-.009
KRESGE	0.074	-.131	-.042	0.046	-.052
LOEWS CORP.	0.020	-.060	0.011	0.097	-.049

* Positively autocorrelated
** Negatively autocorrelated

Appendix I (Part 3)

SERIAL CORRELATION COEFFICIENTS
FOR DAILY RETURNS FOR LAG
INTERVALS K=1-5

	LAG INTERVALS (DAYS)				
COMMON STOCK (CONTINUED):	1	2	3	4	5
MC DONALDS'	0.076	-.111	0.024	0.074	0.043
MERCK	0.099	-.072	-.039	-.020	0.103
MONSANTO *	0.257	-.047	0.013	-.023	0.010
MOTOROLA	0.132	-.130	-.120	-.012	0.024
NORTHWEST AIRLINES	-.022	-.110	0.052	-.068	-.077
PENNZOIL	0.101	-.127	-.118	-.026	0.129
PFIZER CORP.	0.028	-.066	0.018	0.050	0.089
PHELPS DODGE	0.126	-.029	-.066	-.017	-.032
PHILLIP MORRIS	0.075	-.132	-.003	0.054	-.014
POLAROID **	-.077	-.108	-.080	0.081	-.001
PROCTER & GAMBLE	0.083	-.045	0.011	-.082	-.042
R.C.A.	-.031	-.031	-.087	0.017	0.155
SEARS	0.198	-.052	-.056	-.045	-.003
SPERRY RAND	0.062	-.117	-.003	0.075	0.075
TEXACO	0.088	-.010	-.185	-.041	-.046
TEXAS INSTRUMENTS *	0.151	-.005	0.048	-.003	0.016
UNION CARBIDE	0.156	-.226	-.086	0.080	-.001
U.S. STEEL	0.171	-.055	-.009	0.003	-.066
UPJOHN	0.058	-.056	-.003	0.031	0.056

* Positively autocorrelated
** Negatively autocorrelated

Appendix II

Description of the Holding Period Problem

Even a researcher who has never heard of the "holding period problem" (sometimes referred to as the intervaling or horizon problem) will usually make an implicit holding period assumption when he does empirical research. Financial research usually involves flow variables (as opposed to stock variables) and flow variables are only meaningful if the interval over which they are measured is specified (e.g., monthly returns; yearly sales).

Obviously, the returns, expected returns and variances of a security which is held for one day differ substantially from the returns, expected returns and variances of a security which is held for one year. The efficient frontier is nothing but a graphical description of optimum expected return and variance combinations. If expected returns and variances are different for different holding period assumptions (one day? one year?) it is not surprising that efficient frontier composition and shape may also be sensitive to the holding period length assumption.

Obviously, one day variances and covariances differ in magnitude from variances and covariances measured over a one year holding period. Beta is nothing more than a covariance divided by a variance. It should therefore come as no great surprise that beta can change from one holding period length assumption to another.

These holding period dependent changes in beta and the efficient frontier have a number of important implications:

If the composition of the efficient frontier changes from one holding period assumption to another (as suggested by this paper) we are faced with the practical inconvenience of having to create a different efficient frontier for every different holding period length we may be interested in.

An efficient frontier based upon daily data will give a listing of the portfolios which offer the lowest one day variance at every given level of one day expected return. Unfortunately, totally different portfolios may be necessary to minimize one year variance for every level of one year expected return.

In 1965 Tobin demonstrated that under the assumptions of ·tationarity and independence, the composition of the efficient frontier does not change from one holding period length assumption to another. But, the shape of the efficient frontier does change. Gressis, Philipatos and Hayya (1976) have pointed out that as the shape of the efficient frontier changes from one holding period length assumption to another, the capital market line intersects the efficient frontier at a different point. GPH then point out that the market portfolio will be the average of these different efficient portfolios.

In 1969 Jensen pointed out that the capital market relationship between security beta and security expected return can only be linear for one specific holding period assumption. Levhari and Levy (1977) have expanded upon this analysis and have demonstrated a bias in Treynor's (1965) reward to volatility ratio. They have shown that a security with a high expected return relative to its beta for one holding period assumption can have a low expected return relative to its beta for some other

holding period assumption. Levy (1972) has demonstrated a similar problem with Sharpe's (1966) reward to variability ratio.

One source of confusion about the holding period assumption is the fact that there are usually two time dimensions involved in empirical research: the interval between successive measurements of the flow variables (e.g., monthly returns) and the total interval from which measurements were taken (e.g., five years of monthly returns).

When we use five years of monthly data to calculate expected returns or variances, we have not analysed the effect of holding a security for five years. We have analysed the effect of holding a security for one month using 60 one month observations.

FOOTNOTES

*This paper draws on my dissertation submitted to the Faculty of the Graduate School of Business Administration of the University of Michigan. I would particularly like to thank my Dissertation Chairman, Timothy J. Nantell, for his invaluable assistance.

[1]Similar time/variance/autocorrelation equations were derived independently by the author during the course of his disertation.

[2]"Intervaling length" and "holding period length" are similar concepts. The "intervaling length" is the length of the time interval between successive measurements of a phenomenon. The "holding period length" is the length of time an investor wishes to hold an investment.

When measurements based on a given intervaling length are used to predict the behavior of an investment, the "intervaling length" can be referred to as the "holding period length."

Throughout most of this paper we will use the term "holding period length" to emphasize the practical implications of the "interval length" assumption.

[3]Most multiperiod model research has assumed serial independence of security returns (see, for example, Chen, Jen and Zionts (1971), Elton and Gruber (1974) or Mossin (1968)). Our relaxation of the independence assumption in the single period case probably has interesting implications for the multiperiod case also.

[4]The theoretical derivations of N period portfolio expected returns and variances (equations (1) through (12)) assume stationarity of <u>portfolio</u> returns. Some researchers prefer to assume <u>security</u> return stationarity in which case <u>portfolio</u> stationarity will (in general) require rebalancing at the end of each of the N periods.

Given the use of daily data and a maximum compounding interval of 24 days, the effect of portfolio rebalancing will be relatively small.

[5]The use of log returns simplifies our mathematical derivations. These log returns are actually continuously compounded returns. These log returns may be viewed as approximations of returns compounded over a finite time interval.

[6]Our empirical work indicates that the second term on the right side of equation (12) is slightly (but erratically) effected by autocorrelation. The second term seems to be a positive function of autocorrelation. N period variance is therefore <u>more</u> strongly effected by autocorrelation than is suggested by the first term alone.

[7]The primary source of our data was a computer tape generously supplied by Drexel Burnham and Company. The 78 securities we used in our research include all of the securities on the tape for which there was no missing data.

[8]Articles by Praetz (1972) and Blattberg and Gonedes (1974) suggest that daily return data is reasonably well approximated by a t distribution with a degree of freedom parameter ranging from about 3 to about 5. This suggests that our statistical tests of autocorrelation will understate the true variability of short time horizon returns.

[9]The levels of statistically significant autocorrelation we observed in our data were generally comparable to the findings of other researchers.

We found that 38.5 percent and 14.1 percent of our data base exhibited first order daily autocorrelation significant at the 5 percent and 1 percent levels respectively. Corresponding figures for Fama's (1965) study of the Dow Jones 30 Industrials were 36.6 percent and 26.6 percent respectively. The corresponding figures for Leabo and Rogalski's (1975) study of American and NYSE warrants were 50 percent and 39.6 percent.

[10]The time/variance tests of Osborne (1959) and Granger and Morgenstern (1970) provide little or no evidence of security autocorrelation. The time/variance tests of Young (1971) and Leabo and Rogalski (1975) and our own research as described in Section IV of this paper seem to indicate significant amounts of autocorrelation.

This difference probably results from the fact that Osborne and Granger and Morgenstern looked at the average time/variance relationship of a group of securities. The average time/variance relationship of our own data base seems to indicate no significant autocorrelation. However, Table 1 indicates that (in our own work at least) this lack of an average relationship results from the cancelling out of large positive and negative serial correlation effects. Young's study of individual security time/variance relationships is generally consistent with our own.

Leabo and Rogalski study average time/variance relationships but the warrants they studied exhibited so much negative autocorrelation that even these average relationships suggest a substantial deviation from a linear time/variance relationship.

[11]Young points out that if we have two series, x_t and y_t, $t=1, 2, \ldots, T$, that are normally distributed with a correlation coefficient $\rho_{x,y}$, and if:

$$\lambda = \sigma_x/\sigma_y,$$

$$u = x + \lambda y,$$

$$v = x - \lambda y;$$

then testing

$$H_o:\ \sigma_x^2 = \lambda^2\sigma_y^2, \qquad\qquad H_o:\ \rho_{uv} = 0,$$

is equivalent to

$$H_a:\ \sigma_x^2 \neq \lambda^2\sigma_y^2, \qquad\qquad H_a:\ \rho_{uv} \neq 0.$$

The test of significance is made by comparing the correlation coefficient, $r_{u,v}$, to Student's t distribution with T-2 degrees of freedom.

[12]Efficient portfolios were generated using an algorithm derived by Alexander (1976).

[13]The entries listed in row 9 ("Minimum Variance") of Tables 2 and 3 do not correspond to benchmark portfolios. Row 9 of Table 2 lists the expected returns and variances of the actual optimum minimum variance portfolios for each of the stated holding period assumptions. Row 9 of Table 3 lists the composition of these actual minimum variance portfolios.

[14]The reader should recall that our short and long holding period efficient frontiers were calculated using the exact same data over the exact same time period. Different holding period length assumptions were created by using different groupings of the same data.

The changes in variance and efficient frontier composition we observe are therefore due to changes in the holding period length assumption. No new data is involved.

[15]These changes in relative variance and efficiency have another interesting implication: Our traditional concept of the "risk-return trade off" may have to be modified. Table 2 identifies several portfolios which are low risk and low return (and efficient) for one holding period length assumption but high risk low return (and inefficient) for a different holding period assumption. This suggests that some portfolios cannot be uniquely classified as high risk or low risk. The same portfolio may fit into either classification depending upon the holding period length assumption.

[16]Sarnat suggests in a footnote that his findings may be due to the holding period assumption. He suggests that fixed income security representation would be increased for longer holding period assumptions. This paper suggests that shorter assumptions may also increase representation.

[17]This holding period effect on the representation of securities in efficient frontier portfolios may provide an interesting test of the length of the market holding period. We could try to determine which holding period length assumption causes the weighting of securities in the efficient frontier to most closely resemble a market value weighting. By this reasoning, the relative findings of Alexander and Sarnat and Table III of this paper seem to support the idea first proposed by Jensen (1969) that the market horizon is relatively short.

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